Which self-maps appear as lattice endomorphisms?

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Abstract. Let $f : A \to A$ be a self-map of the set A. We give a necessary and sufficient condition for the existence of a lattice structure $(A, \vee, \wedge)$ on A such that f becomes a lattice endomorphism with respect to this structure.

1. INTRODUCTION

A *partially ordered set (poset)* is a set P together with a reflexive, antisymmetric, and transitive (binary) relation $r \subseteq P \times P$. For $(x, y) \in r$ we write $x \leq_r y$ or simply $x \leq y$. If $r \subseteq r' \subseteq P \times P$ for the partial orders r and r', then r' is an *extension* of r. A map $p : P \longrightarrow P$ is *order-preserving* if $x \leq y$ implies $p(x) \leq p(y)$ for all $x, y \in P$. The poset $(P, \leq)$ is a *lattice* if any two elements $x, y \in P$ have a unique least upper bound (lub) $x \vee y$ and a unique greatest lower bound (glb) $x \wedge y$ (in P). The operations $\vee$ and $\wedge$ are associative, commutative, and satisfy the following absorption laws: $(x \vee y) \wedge y = y$ and $(x \wedge y) \vee y = y$. Any binary operations $\vee$ and $\wedge$ on P having these properties define a binary relation $r = \{(x, x \vee y) : x, y \in P\} \subseteq P \times P$ on P, which is a partial order. In fact $(P, \leq_r)$ is a lattice with lub $\vee$ and glb $\wedge$. Lattices play a fundamental role in many areas of mathematics (see [1],[3]).

In the present paper we consider a self-map $f : A \longrightarrow A$ of a set A. A list $x_1, \ldots, x_n$ of distinct elements from A is a *cycle* (of *length* n) with respect to f if $f(x_i) = x_{i+1}$ for each $1 \leq i \leq n-1$ and also $f(x_n) = x_1$. A *fixed point* of the function f is a cycle of length 1, i.e. an element $x_1 \in A$ with $f(x_1) = x_1$. A cycle that is not a fixed point is *proper.*

If $(A, \vee, \wedge)$ is a lattice (on the set A) such that $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in A$, then f is a *lattice endomorphism* of $(A, \vee, \wedge)$. A lattice endomorphism is an order-preserving map (with respect to the order relation of the lattice), but the converse is not true in general. For a proper cycle $x_1, \ldots, x_n \in A$ with respect to a lattice endomorphism f, if we put

$$p = x_1 \vee x_2 \vee \cdots \vee x_n$$

and

$$q = x_1 \wedge x_2 \wedge \cdots \wedge x_n,$$

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then $p \neq q$. The equalities

$$f(p) = f(x_1) \vee f(x_2) \vee \cdots \vee f(x_n) = x_2 \vee \cdots \vee x_n \vee x_1 = p,$$
$$f(q) = f(x_1) \wedge f(x_2) \wedge \cdots \wedge f(x_n) = x_2 \wedge \cdots \wedge x_n \wedge x_1 = q$$

show that p and q are distinct fixed points of f. It follows that any lattice endomorphism having a proper cycle must have at least two fixed points.

We prove that the above combinatorial property completely characterizes the possible lattice endomorphisms. More precisely, for a map $f : A \longrightarrow A$ there exists a lattice $(A, \vee, \wedge)$ on A such that f is a lattice endomorphism of $(A, \vee, \wedge)$ if and only if f has no proper cycles or f has at least two fixed points.

The construction in the proof of our main result is based on the use of the maximal f-compatible extensions of an f-compatible partial order relation on A. Such extensions were completely determined in [2] and [5]. In order to make the exposition more self-contained, we present the necessary background about maximal compatible extensions.

2. PRELIMINARY DEFINITIONS AND RESULTS

Let $f : A \longrightarrow A$ be a function, and define the equivalence relation $\sim_f$ as follows: for $x, y \in A$, let $x \sim_f y$ if $f^k(x) = f^l(y)$ for some integers $k \geq 0$ and $l \geq 0$. The equivalence class $[x]_f$ of an element $x \in A$ is the *f-component* of x. We note that $[x]_f$ is closed with respect to the action of f and hence contains the *f-orbit* of x:

$$\{x, f(x), \ldots, f^k(x), \ldots\} \subseteq [x]_f \ .$$

An element $c \in A$ is *cyclic* with respect to f if $f^m(c) = c$ for some integer $m \geq 1$. The *period* of a cyclic element c, written as $n(c)$, is defined by

$$n(c) = \min\{m : m \geq 1 \text{ and } f^m(c) = c\},$$

and $f^k(c) = f^l(c)$ holds if and only if $k - l$ is divisible by n. The *full cycle* of a cyclic element c is the f-orbit $\{c, f(c), ..., f^{n(c)-1}(c)\}$. The f-orbit of x is finite if and only if $[x]_f$ contains a cyclic element. If $c_1, c_2 \in [x]_f$ are cyclic elements, then $n(c_1) = n(c_2) = n(x)$, and this number is the *period* of x. If the f-orbit of x is infinite, then put $n(x) = \infty$. Clearly, $x \sim_f y$ implies $n(x) = n(y)$. We note that the presence of a cyclic element in $[x]_f$ does not imply that $[x]_f$ is finite. The function f has a proper cycle if there exists a cyclic element $c \in A$ with respect to f such that $n(c) \geq 2$.

2.1. Theorem (see [4]). *Let r be an order relation on the set A, and let $f : A \to A$ be an order-preserving map with respect to r. If there is no proper cycle of f, then there exists a linear extension R of r such that f is order-preserving with respect to R.*

2.2. Corollary. *If $f : A \longrightarrow A$ is a function with no proper cycles, then there exists a distributive lattice $(A, \vee, \wedge)$ on A such that f is a lattice endomorphism of $(A, \vee, \wedge)$.*

The following definitions appear in [2]. A pair $(x, y) \in A \times A$ is *f-prohibited* if there exist integers k, l, and n with $k \geq 0$, $l \geq 0$, and $n \geq 2$ such that n is not a divisor of $k - l$, the elements $f^k(x), f^{k+1}(x), ..., f^{k+n-1}(x)$ are distinct and $f^{k+n}(x) = f^k(x) = f^l(y)$. For an f-prohibited pair (x, y) and integers k and n as

above, $f^k(x)$ is a cyclic element in $[x]_f = [y]_f$ of period n. The *distance* $d(y,c)$ between an element $y \in [x]_f$ and a given cyclic element $c \in [x]_f$ (of period $n \geq 1$) is defined by

$$d(y,c) = \min\{t : t \geq 0 \text{ and } f^t(y) = c\}.$$

Clearly, $f^t(y) = c$ holds if and only if $t \geq d(y,c)$ and $t - d(y,c)$ is divisible by n. We note that $d(f(c),c) = n-1$, and for $y \neq c$ we have $d(f(y),c) = d(y,c) - 1$. It is straightforward to see that (x,y) is f-prohibited if and only if $d(x,c) - d(y,c)$ is not divisible by n.

2.3. Proposition (see [2]). *Let r be an order relation on the set A and $f : A \to A$ be an order-preserving map with respect to r. If $(x,y) \in A \times A$ is an f-prohibited pair, then x and y are incomparable with respect to r.*

2.4. Lemma (see [2]). *Let $f : A \longrightarrow A$ be a self-map on a set A. Let c be a cyclic element, with $c \in [x]_f$ for some $x \in A$. If r is an order relation on A, and f is order-preserving with respect to r, then there exists an order relation ρ on $[x]_f$ with the following properties:*
1. *ρ is an extension of r (on $[x]_f$): $r \cap ([x]_f \times [x]_f) \subseteq \rho$,*
2. *f is order-preserving with respect to ρ,*
3. *$[x]_f$ is the disjoint union of sets $E_0, \ldots, E_{n-1}$ and each*

$$E_i = \{u \in [x]_f : d(u,c) - i \text{ is divisible by } n(c)\}\ ,\ 0 \leq i \leq n-1$$

   is a chain with respect to ρ,
4. *$f(E_0) \subseteq E_{n-1}$ and $f(E_i) \subseteq E_{i-1}$ for $1 \leq i \leq n-1$,*
5. *any element $(u,v) \in E_i \times E_j$ with $i \neq j$ is an f-prohibited pair, and the set $\{u,v\}$ has no upper and lower bounds in $[x]_f$ with respect to ρ.*

3. MAKING f A LATTICE ENDOMORPHISM

3.1. Theorem. *Let r be an order relation on the set A, and let $f : A \longrightarrow A$ be an order-preserving map with respect to r having distinct fixed points $p, q \in A$. If x and y are r-incomparable for all $x, y \in A$ such that $[x]_f \neq [y]_f$ and $2 \leq n(x) \neq \infty$, then there exists an extension R of r such that $(A, \leq_R)$ is a lattice and f is a lattice endomorphism of $(A, \leq_R)$.*

Proof. Let

$$A_0 = \{x \in A : [x]_f \text{ contains a proper cycle}\} = \{x \in A : 2 \leq n(x) \neq \infty\}$$

The set A_0 is the f-cyclic part of A. Let

$$A_* = A \setminus A_0 = \{x \in A : [x]_f \text{ has no proper cycle}\} = \{x \in A : n(x) = 1 \text{ or } n(x) = \infty\}$$

The set A_* is the f-acyclic part of A. We have either $[x]_f \subseteq A_0$ or $[x]_f \subseteq A_*$ for all $x \in A$. Clearly, both A_0 and A_* are closed with respect to the action of f, i.e. $f(A_0) \subseteq A_0$ and $f(A_*) \subseteq A_*$. Since $f : A_* \longrightarrow A_*$ has no proper cycle (in A_*), Theorem 2.1 ensures the existence of a linear extension R_* of $r \cap (A_* \times A_*)$ (on A_*) such that f is order-preserving with respect to R_*. In view of $p, q \in A_*$, we may assume $p \leq_{R_*} q$.

For an appropriate subset $\{x_t : t \in T\}$ of A_0, where the indices are taken from an idex set T, we have $\{[x]_f : x \in A_0\} = \{[x_t]_f : t \in T\}$, and $[x_t]_f \neq [x_s]_f$ for

all $t, s \in T$ with $t \neq s$. Such a subset $\{x_t : t \in T\} \subseteq A_0$ is an *irredundant set of representatives* of the equivalence classes of $\sim_f$ (in A_0). That is

$$A_0 = \bigcup_{t \in T} [x_t]_f \text{ and } [x_t]_f \cap [x_s]_f = \varnothing \text{ for all } t, s \in T \text{ with } t \neq s.$$

Call two elements of A *concurrent* if some power of f maps them to the same element. Concurrency is an equivalence relation finer than $\sim_f$. For $t \in T$, the $\sim_f$-class of x_t is partitioned into $n(x_t)$ concurrency classes:

$$[x_t]_f = E_0^{(t)} \cup E_1^{(t)} \cup \ldots \cup E_{n(x_t)-1}^{(t)}, \text{ where}$$

$$E_i^{(t)} = \{u \in [x_t]_f : d(u, c) - i \text{ is divisible by } n(x_t)\}$$

for some fixed cyclic element $c \in [x_t]_f$. Application of Lemma 2.4 gives the existence of a partial order extension ρ_t of r on $[x_t]_f$ ($r \cap ([x_t]_f \times [x_t]_f) \subseteq \rho_t$ holds) such that f preserves ρ_t and each $E_i^{(t)}$ is a chain with respect to ρ_t.

Take the following subsets of $A \times A$:

$$P = \{(a, x) : a \in A_*, x \in A_0 \text{ and } a \leq_{R_*} p\} \text{ and } Q = \{(y, b) : b \in A_*, y \in A_0 \text{ and } q \leq_{R_*} b\}.$$

Let

$$R = R_* \cup \left(\bigcup_{t \in T} \rho_t\right) \cup P \cup Q.$$

We claim that R is an extension of r that is a lattice and that f is a lattice endomorphism of $(A, \leq_R, \vee, \wedge)$. The proof consists of the following straightforward steps.

Notice that $R_* \subseteq A_* \times A_*$, $\rho_t \subseteq [x_t]_f \times [x_t]_f \subseteq A_0 \times A_0$, $P \subseteq A_* \times A_0$, and $Q \subseteq A_0 \times A_*$. Also the direct products $A_* \times A_*$, $A_* \times A_0$, $A_0 \times A_*$, and $[x_t]_f \times [x_t]_f$ (for $t \in T$) are pairwise disjoint.

In order to see $r \subseteq R$, take $(u, v) \in r$.
(1) If $(u, v) \in A_* \times A_*$, then $r \cap (A_* \times A_*) \subseteq R_*$ implies $(u, v) \in R$.
(2) If $(u, v) \in A_* \times A_0$, then $[u]_f \neq [v]_f$ and $2 \leq n(v) \neq \infty$ contradicts $(u, v) \in r$.
(3) $(u, v) \in A_0 \times A_*$ is also impossible.
(4) If $(u, v) \in A_0 \times A_0$, then $(u, v) \in [x_t]_f \times [x_s]_f$ for some $t, s \in T$. Clearly, $t \neq s$ would imply $[u]_f \neq [v]_f$, and then $2 \leq n(u) \neq \infty$ contradicts $(u, v) \in r$. Thus $t = s$, and $r \cap ([x_t]_f \times [x_t]_f) \subseteq \rho_t$ yields $(u, v) \in R$.

We prove that R is a partial order.
Antisymmetry: Let $(u, v) \in R$ and $(v, u) \in R$.
(1) If $(u, v), (v, u) \in R_*$, then $u = v$ follows from the antisymmetric property of R_*
(2) If $(u, v) \in \rho_t$ and $(v, u) \in \rho_s$, then $t = s$, and $u = v$ follows from the antisymmetric property of ρ_t.
(3) If $(u, v) \in P$ and $(v, u) \in Q$, then $u \leq_{R_*} p$ and $q \leq_{R_*} u$ imply $q \leq_{R_*} p$, contradicting with $p \leq_{R_*} q$ and $p \neq q$.
(4) If $(u, v) \in Q$ and $(v, u) \in P$, then interchanging the roles of u and v leads to a similar contradiction as in case (3).
Transitivity: Let $(u, v) \in R$ and $(v, w) \in R$.
(1) If $(u, v), (v, w) \in R_*$, then $(u, w) \in R_*$ follows from the transitivity of R_*.
(2) If $(u, v) \in R_*$ and $(v, w) \in P$, then $u \leq_{R_*} v \leq_{R_*} p$ and $w \in A_0$ imply $(u, w) \in P$.
(3) If $(u, v) \in \rho_t$ and $(v, w) \in \rho_s$, then we have $t = s$, and $(u, w) \in \rho_t$ follows from the transitivity of ρ_t.
(4) If $(u, v) \in \rho_t$ and $(v, w) \in Q$, then $u, v \in A_0$, $w \in A_*$, and $q \leq_{R_*} w$. It follows that $(u, w) \in Q$.

(5) If $(u,v) \in P$ and $(v,w) \in \rho_t$, then $v,w \in A_0$, $u \in A_*$, and $u \leq_{R_*} p$. It follows that $(u,w) \in P$.
(6) If $(u,v) \in P$ and $(v,w) \in Q$, then $u \leq_{R_*} p \leq_{R_*} q \leq_{R_*} w$, from which $(u,w) \in R_*$ follows.
(7) If $(u,v) \in Q$ and $(v,w) \in R_*$, then $u \in A_0$ and $q \leq_{R_*} v \leq_{R_*} w$ imply $(u,w) \in P$.
(8) If $(u,v) \in Q$ and $(v,w) \in P$, then $q \leq_{R_*} v \leq_{R_*} p$ contradicts $p \leq_{R_*} q$ and $p \neq q$.

We note that f is order-preserving with respect to $(A_*, \leq_{R_*})$, and $([x_t]_f, \rho_t)$ for $t \in T$. In order to check the order-preserving property of f with respect to $(A, \leq_R)$, it is enough to see that $(a,x) \in P$ implies $(f(a), f(x)) \in P$ and $(y,b) \in Q$ implies $(f(y), f(b)) \in Q$. Obviously, $a \in A_*$, $x \in A_0$, and $a \leq_{R_*} p$ imply $f(a) \in A_*$, $f(x) \in A_0$, and $f(a) \leq_{R_*} f(p) = p$. Similarly, $b \in A_*$, $y \in A_0$, and $q \leq_{R_*} b$ imply $f(b) \in A_*$, $f(y) \in A_0$, and $q = f(q) \leq_{R_*} f(b)$.

If $u,v \in A$ are comparable elements with respect to R, then the existence of the supremum $u \vee v$ and the infimum $u \wedge v$ in $(A, \leq_R)$ is evident; moreover, the order-preserving property of f ensures that

$$f(u \vee v) = f(u) \vee f(v) \text{ and } f(u \wedge v) = f(u) \wedge f(v).$$

If $u,v \in A$ are incomparable elements with respect to R, then we have the following possibilities.
(1) If $u \in A_*$ and $v \in A_0$, then $(u,v) \notin P$, $(v,u) \notin Q$, and the linearity of R_* imply $p \leq_{R_*} u \leq_{R_*} q$, from which $u \vee v = q$ and $u \wedge v = p$ follow in $(A, \leq_R)$. Since $f(u) \in A_*$, $f(v) \in A_0$, and $p = f(p) \leq_{R_*} f(u) \leq_{R_*} f(q) = q$, we deduce that

$$f(u \vee v) = f(q) = q = f(u) \vee f(v) \text{ and } f(u \wedge v) = f(p) = p = f(u) \wedge f(v).$$

(2) If $u \in A_0$ and $v \in A_*$, then interchanging the roles of u and v leads to the same result as in case (1).
(3) If $u,v \in A_0$ and $[u]_f \neq [v]_f$, then $u \vee v = q$ and $u \wedge v = p$ in $(A, \leq_R)$ follow directly from the definition of R. Since $f(u), f(v) \in A_0$, and $[f(u)]_f = [u]_f \neq [v]_f = [f(v)]_f$, we deduce

$$f(u \vee v) = f(q) = q = f(u) \vee f(v) \text{ and } f(u \wedge v) = f(p) = p = f(u) \wedge f(v).$$

(4) If $u,v \in A_0$ and $[u]_f = [v]_f = [x_t]_f$ for some unique $t \in T$, then $(u,v) \notin \rho_t$ implies $(u,v) \in E_i^{(t)} \times E_j^{(t)}$ for some unique $0 \leq i,j \leq n(x_t) - 1$ with $i \neq j$. In view of $E_i^{(t)} \cap E_j^{(t)} = \varnothing$ and (5) of Lemma 2.4, we conclude that the set $\{u,v\}$ has no upper and lower bounds in $([x_t]_f, \rho_t)$. It follows that $u \vee v = q$ and $u \wedge v = p$ in $(A, \leq_R)$.

Since $f(E_i) \subseteq E_{i-1}$ implies $(f(u), f(v)) \in E_{i-1}^{(t)} \times E_{j-1}^{(t)}$ (notice that $E_{-1}^{(t)} = E_{n(x_t)-1}^{(t)}$), we deduce in a similar way

$$f(u) \vee f(v) = q = f(q) = f(u \vee v) \text{ and } f(u) \wedge f(v) = p = f(p) = f(u \wedge v). \square$$

3.2. Corollary. *If the number of fixed points of the function $f : A \longrightarrow A$ is at least 2, then there exists a lattice structure $(A, \vee, \wedge)$ on A such that f is a lattice endomorphism of $(A, \vee, \wedge)$.*

Proof. Let p and q be distinct fixed points of f. The application of Theorem 3.1 yields a partial order extension R of the identity partial order $\{(x,x) : x \in A\}$ such that $(A, \leq_R, \vee, \wedge)$ is a lattice and f is a lattice endomorphism of $(A, \leq_R, \vee, \wedge)$. $\square$

The combination of Corollaries 2.2 and 3.2 provides the complete answer (formulated in the introduction) to the question in the title of the paper. We pose a further problem.

3.3. Problem. *Consider an arbitrary function* $f : A \longrightarrow A$. *Find necessary and sufficient conditions for the existence of a modular (or distributive) lattice structure* $(A, \vee, \wedge)$ *on* A *such that* f *becomes a lattice endomorphism of* $(A, \vee, \wedge)$. *The similar question seems to be interesting for other algebraic structures such as (Abelian) groups, rings and modules.*

3.4. Example. Let $A = \{p, q, x_1, x_2, \ldots, x_n\}$, where $n \geq 3$, and let $f : A \longrightarrow A$ be a function with $f(p) = p$, $f(q) = q$, $f(x_n) = x_1$, and $f(x_i) = x_{i+1}$ for $1 \leq i \leq n-1$. If f is an endomorphism of some lattice $(A, \leq, \vee, \wedge)$, then f is order-preserving with respect to $(A, \leq)$, and Proposition 2.3 ensures that the proper cycle $\{x_1, \ldots, x_n\}$ of f is an antichain in $(A, \leq)$. Since $x_1 \vee \cdots \vee x_n$ and $x_1 \wedge \cdots \wedge x_n$ are distinct fixed points of f, one of $x_1 \vee \cdots \vee x_n$ and $x_1 \wedge \cdots \wedge x_n$ is p and the other is q. Thus $(A, \leq, \vee, \wedge)$ is isomorphic to the lattice M_n in both cases. It follows that there is no distributive lattice structure on A making f a lattice endomorphism (even though f has two fixed points).

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